SECTION 17.8: THE DIVERGENCE THEOREM

RECALL: Green's Theorem, Flux Form: under suitable conditions: $\oint_C \vec{F} \cdot \hat{N}\, ds = \int\int_R \nabla \cdot \vec{F}\, dA$

THE DIVERGENCE THEOREM: Under suitable conditions: $\oiint_S \vec{F} \cdot \hat{N}\, dS = \int\int\int_Q \nabla \cdot \vec{F}\, dV$

EXAMPLE 1: Let $\vec{F}(x, y, z) = \langle 2x, z, 3y \rangle$. Let Q be portion of Quadrant I bounded by $3x + 2y + z = 6$.

1. Calculate the outward flux of $\vec{F}$ across the boundary of Q using flux integrals.

Ans: The flux integrals total to: $33 + (-9) + 0 + (-12) = 12$.

2. Check your answer using the Divergence Theorem by calculating the corresponding triple integral.

Ans: integrating $(\nabla \cdot \vec{F})(x, y, z) = 2$ over Q gives 12 ✓

EXAMPLE 2: Let $\vec{F}(x, y, z) = \langle 2x, z, 3y \rangle$. (Note: This is the same field from Example 1.)

Let $Q = \{(x, y, z) : 1 \leq x^2 + y^2 + z^2 \leq 4\}$.

1. Calculate the outward flux of $\vec{F}$ across the boundary of Q using flux integrals.

Ans: The flux integrals total to: $\dfrac{64\pi}{3} + \left(-\dfrac{8\pi}{3}\right) = \dfrac{56\pi}{3}$.

2. Check your answer using the Divergence Theorem by calculating the corresponding triple integral.

Ans: integrating $(\nabla \cdot \vec{F})(x, y, z) = 2$ over Q gives $\dfrac{56\pi}{3}$ ✓

EXAMPLE 3: Let $\vec{F}$ be an inverse square field: for $k > 0$:

$$\vec{F}(x,y,z) = \left\langle \frac{k\,x}{(x^2+y^2+z^2)^{3/2}}, \frac{k\,y}{(x^2+y^2+z^2)^{3/2}}, \frac{k\,z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

Let S be a sphere of radius R centered at the origin and let Q be the solid bounded by S.

1. Why can't we use the Divergence Theorem to find the outward flux of $\vec{F}$ across the boundary of Q?

2. Find the flux of $\vec{F}$ across S using a surface integral.

Ans: The flux integral gives: $4\pi k$.

3. Calculate $\displaystyle\iiint_W \nabla \cdot \vec{F}\, dV$ where $W = \{(x,y,z) : r^2 \leq x^2+y^2+z^2 \leq R^2\}$ and take a limit as $r \to 0^+$.

HOMEWORK: Section 17.8: 9 - 49 every other odd.